Non-singular developable triangular Bézier patches

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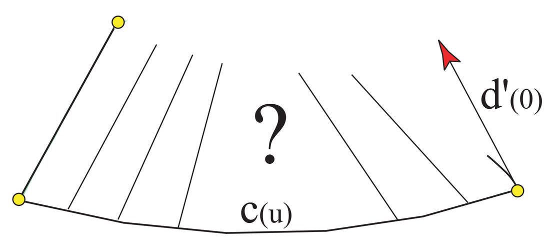
Summary

- We show a characterisation of developable surfaces in the form of Bézier triangular patches.
- Constructions used for rectangular patches are not useful, since they produce degenerate triangular patches.
- Explicit constructions of non-degenerate developable triangular patches are provided.
- Interpolation of a developable triangle between a curve $c(u)$, the last ruling and initial velocity of the other bounding curve $d(u)$.



Previous work

- Construction of developable surfaces of low degree [7-8].
- Construction of developable surfaces in dual space [3,9].
- Constructions grounded on the De Casteljau algorithm [1-2,4-6].

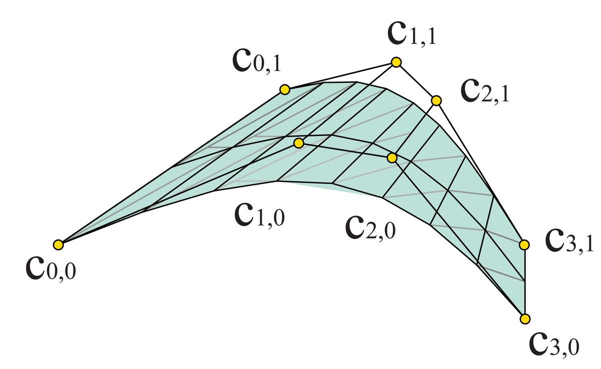
1

Introduction

- Cones and cylinders are easily represented as NURBS surfaces.
- On the contrary, more general developable surfaces are difficult to construct due to the vanishing gaussian curvature condition.
- Solutions for Bézier [1-2], spline [4] and rational [5] developable rectangular patches have been found.
- The simplest require linear relations between the vertices of the control net [1].
- For instance, for Bézier rectangular patches, the control polygons of the bounding curves, $\{c_0,\dots,c_n\}$, $\{d_0,\dots,d_n\}$, are related by

$$(1-\Lambda)c_j+\Lambda c_{j+1}=(1-M)d_j+Md_{j+1},\quad j=0,\dots,n-1,$$

for fixed Λ, M.



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Triangular ruled surfaces

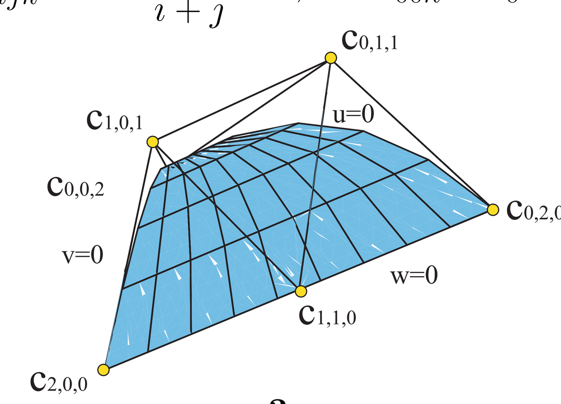
- A triangular patch of degree n is defined as

$$c(u,v,w)=\sum_{i+j+k=n}\frac{n!}{i!j!k!}u^iv^jw^kc_{ijk},\quad u+v+w=1,$$

for a control net $\left\{c_{ijk}:\ i+j+k=n\right\}$ of $(n+2)(n+1)/2$ vertices.

- The boundary of the patch is formed by the curves at $v=0$, $u=0$, $w=0$.
- We wish a triangular ruled patch between the curves at $v=0$ and $u=0$, with control polygons $\{c_0,\dots,c_n\}$ and $\{d_0,\dots,d_n\}$.
- By demanding that constant w curves be straight lines we get

$$c_{ijk}=\frac{ic_{i+j}+jd_{i+j}}{i+j},\qquad c_{00n}=c_0=d_0.$$



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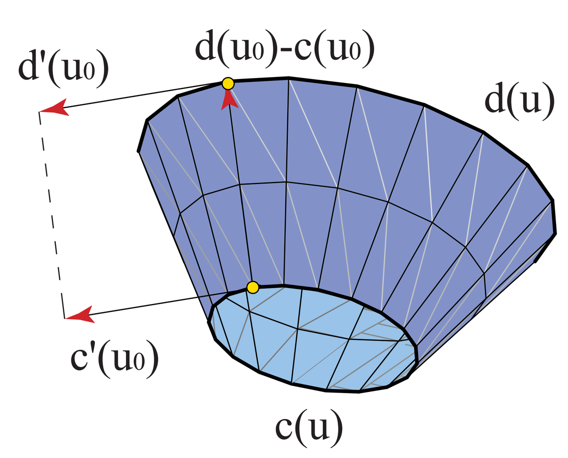
Developable Bézier surfaces

- A developable surface bounded by two curves $c(u)$, $d(u)$ has the same tangent plane for all points of each ruling,

$$d'(u)\cdot c'(u)\times\big(d(u)-c(u)\big)=0.$$



- This may be solved in terms of blossoms [4],

$$c[u^{<n-1>},\Lambda(u)]=d[u^{<n-1>},M(u)],$$

for some functions $\Lambda(u)$, $M(u)$.

- The constant Λ, M case is the one shown in **2**.

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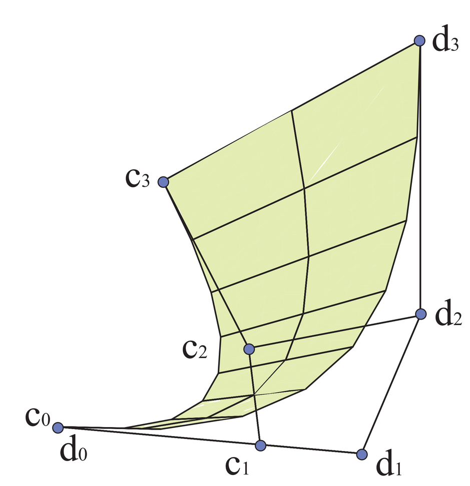
Degenerate triangular patches

- Unfortunately, the constant case produces just degenerate triangular patches,

$$(1-\Lambda)c_0+\Lambda c_1=(1-M)d_0+Md_1\Rightarrow d_1=\left(1-\frac{\Lambda}{M}\right)c_0+\frac{\Lambda}{M}c_1.$$



- This happens also for general Λ, M unless $\Lambda(0)=0=M(0)$.
- The problem is that for $c(u,v)=(1-v)\,c(u)+v\,d(u)$, $d'(0)$ is parallel to $c'(0)$ and therefore to $c_u(0,v)$ for all v.

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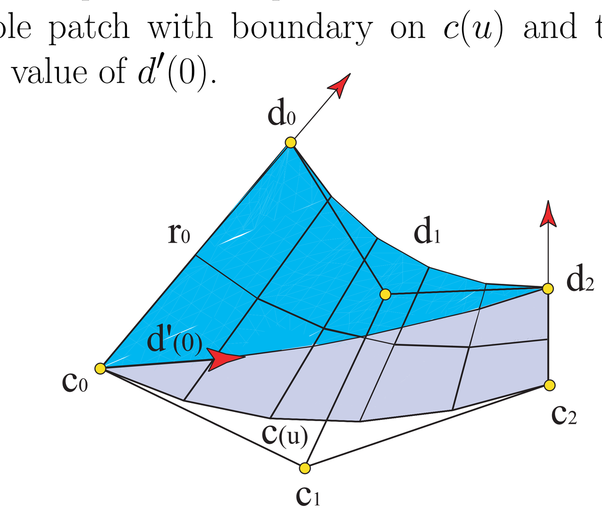
Non-degenerate triangular patches

- We start with a curve $c(u)$ with control polygon $\{c_0,\dots,c_n\}$ and the rulings at both ends, the lines r_0 and r_n. We may fix one end of the ruling, d_0 or d_n, but not both, with a developable surface of constant Λ, M with such boundaries [1].
- For triangles, we begin with $c(u)$ and d_n and the unknown line r_0 to get a developable patch $c(u,v)=c(u)+v\,\mathbf{v}(u)$, $\mathbf{v}=d-c$.
- We shorten the patch along the rulings, $\tilde{c}(u,v)=c(u)+uv\,\mathbf{v}(u)$.
- And prescribe the velocity $d'(0)=c'(0)+\mathbf{v}(0)=n(c_1-c_0)+(d_0-c_0)$ from which we read the auxiliary vertex d_0.
- Hence, by this procedure it possible to find at least one triangular developable patch with boundary on $c(u)$ and the ruling $\overline{d_nc_n}$ fixing the value of $d'(0)$.



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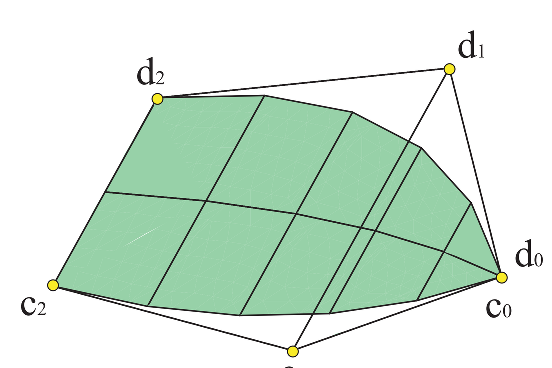
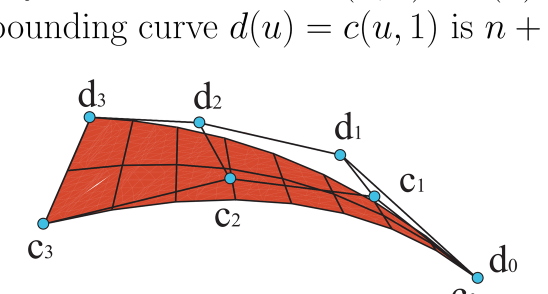
Other triangular developable patches

- Cylinders bounded by a curve $c(u)$ of degree n and generatrices parallel to a constant vector $\mathbf{v}$ are parametrised as $c(u,v)=c(u)+f(u)\,\mathbf{v}$ where $f(u)$ is a polynomial vanishing at $u=0$. The other bounding curve is $d(u)=c(u,1)$.



- Cones bounded by a curve $c(u)$ of degree n and generatrices parallel to $\mathbf{v}(u)$ may be constructed as $c(u,v)=c(u)+u\,\mathbf{v}(u)$. The degree of the bounding curve $d(u)=c(u,1)$ is $n+1$.



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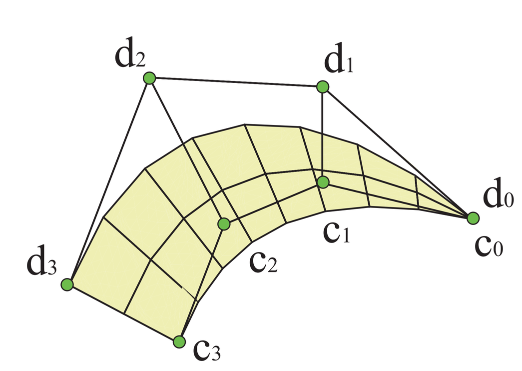
Conclusions

- A way of representing ruled triangular Bézier patches in terms of the bounding curves is produced.
- It is shown that usual constructions for designing general developable Bézier surfaces produce only degenerate Bézier triangular patches.
- A procedure for constructing general non-degenerate developable Bézier triangular surfaces is shown.
- Triangular Bézier cylindrical and conical patches can be also constructed.
- Extension to rational Bézier developable triangular patches is possible.



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